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On Complete Tchebycheff-Systems

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It is shown under what condition a Tchebycheff-space on a real set containing at most one end point, has a Tchebycheff subspace of codimension 1. It is also shown when a complete Tchebycheff-system on a real set M, is also a complete Tchebycheff-system on $\overline{M}$.

1. INTRODUCTION

Zielke [2] proved that an oriented Tchebycheff space on a totally ordered set M, having property (D), has a basis which is a complete Tchebycheff-system on M. This theorem generalizes a theorem of Krein [1] where M is an open interval (a, b). Counterexamples where M is an half-open or a closed interval are given in [2, 3].

In this paper we generalize the theorem of Zielke by weakening property (D), for real sets.

We also give a necessary and sufficient condition for a complete oriented Tchebycheff-system on a real set M to be a complete oriented Tchebycheff system on $\overline{M}$.

2. CONVEXITY CONE

We start by recalling the definitions of a Tchebycheff-system (abbreviated as T-system) and of convexity with respect to a T-system.

DEFINITION 1. Let $\{u_i\}_{i=0}^{n}$ be real-valued functions defined on a real set M. These functions will be called a T-system if the determinant

$$U\begin{pmatrix} u_0, & u_1, \ldots, & u_n \\ t_0, & t_1, \ldots, & t_n \end{pmatrix} = \begin{vmatrix} u_0(t_0) & u_0(t_1) & \cdots & u_0(t_n) \\ u_1(t_0) & u_1(t_1) & \cdots & u_1(t_n) \\ \vdots & & & \vdots \\ u_n(t_0) & u_n(t_1) & \cdots & u_n(t_n) \end{vmatrix} \neq 0 \qquad (1)$$

whenever $t_0 < t_1 < \cdots < t_n$ are in M.

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If M is an interval and the functions are continuous, the determinants (1) have the same sign and we require that they are positive.

The functions $\{u_i\}_{i=0}^n$ will be called complete Tchebycheff-system (abbreviated as CT-system) if $\{u_i\}_{i=0}^k$ is a T-system for all $k = 0, 1,..., n$.

The linear hull of a T-system will be called a T-space.

DEFINITION 2. Let $\{u_i\}_{i=0}^n$ be $n + 1$ continuous function on an interval I, forming a T-system on it. A function f defined on (a, b) = the interior of I is said to be convex with respect to $\{u_i\}_{i=0}^n$ if:

$$U\begin{pmatrix} u_0, u_1, \ldots, u_n, f \\ t_0, t_1, \ldots, t_n, t_{n+1} \end{pmatrix} = \begin{vmatrix} u_0(t_0) & u_0(t_1) & \cdots & u_0(t_{n+1}) \\ u_1(t_0) & u_1(t_1) & \cdots & u_1(t_{n+1}) \\ \vdots & & & \vdots \\ u_n(t_0) & u_n(t_1) & \cdots & u_n(t_{n+1}) \\ f(t_0) & f(t_1) & \cdots & f(t_{n+1}) \end{vmatrix} \geqslant 0 \qquad (2)$$

for all choices of $\{t_i\}_{i=0}^{n+1}$ satisfying $a < t_0 < t_1 < \cdots < t_{n+1} < b$.

The set of the convex functions with respect to $\{u_i\}_{i=0}^u$ on an interval is denoted by $C(u_0, u_1, \ldots, u_n)$ (without referring to the interval if no ambiguity arises).

By $f(a)$ ($f(b)$) we denote the $\lim_{x\to a^+} f(x)$ ($\lim_{x\to b^-} f(x)$) if the limit exists and is finite.

LEMMA 1. *Let $\{u_i\}_{i=0}^n$ be continuous functions on an interval I whose interior is (a, b), forming a T-system on it and let $f \in C(u_0, u_1, \ldots, u_n)$, if:*

$$U\begin{pmatrix} u_0, u_1, \ldots, u_n, f \\ \bar{t}_0, \bar{t}_1, \ldots, \bar{t}_n, \bar{t}_{n+1} \end{pmatrix} = 0 \qquad (3)$$

for some $a \leqslant \bar{t}_0 < \bar{t}_1 < \cdots < \bar{t}_{n+1} \leqslant b$, then there exists a polynomial $u = \sum_{i=0}^n a_i u_i$ such that $f \mid [\bar{t}_0, \bar{t}_{n+1}] = u \mid [\bar{t}_0, \bar{t}_{n+1}]$. (Equalities $\bar{t}_0 = a$ ($\bar{t}_{n+1} = b$), only if $a \in I$ ($b \in I$) and the limit $f(a)$ ($f(b)$) exists).

Proof. Since the functions $u_0, u_1, \ldots, u_n$ are linearly independent, the last row in (3) is a linear combination of the others, i.e.

$$f(\bar{t}_j) = \sum_{i=0}^n a_i u_i(\bar{t}_j), \qquad j = 0, 1, \ldots, n + 1.$$

Define $g = f - \sum_{i=0}^n a_i u_i$, clearly:

$$U\begin{pmatrix} u_0, u_1, \ldots, u_n, g \\ t_0, t_1, \ldots, t_n, t_{n+1} \end{pmatrix} = U\begin{pmatrix} u_0, u_1, \ldots, u_n, f \\ t_0, t_1, \ldots, t_n, t_{n+1} \end{pmatrix}.$$

We show that $g \mid [\bar{t}_0, \bar{t}_{n+1}] = 0$. Let

$$t \in [\bar{t}_0, \bar{t}_{n+1}], \qquad t \neq \bar{t}_j, \qquad j = 0, 1, \ldots, n+1.$$

For some i, $\bar{t}_i < t < \bar{t}_{i+1}$. Now:

$$U\begin{pmatrix} u_0, \ldots, & & & \ldots, u_n, g \\ \bar{t}_0, \ldots, \bar{t}_{i-1}, t, \bar{t}_{i+1}, \ldots, & & & \bar{t}_n, \bar{t}_{n+1} \end{pmatrix}$$
$$= (-1)^{n+i+1} g(t)\, U\begin{pmatrix} u_0, \ldots, & & \ldots, u_n \\ \bar{t}_0, \ldots, \bar{t}_{i-1}, \bar{t}_{i+1}, \ldots, & & \bar{t}_{n+1} \end{pmatrix}$$

and

$$U\begin{pmatrix} u_0, \ldots, & & & \ldots, u_n, g \\ \bar{t}_0, \ldots, \bar{t}_i, t, \bar{t}_{i+2}, \ldots, & & & \bar{t}_n, \bar{t}_{n+1} \end{pmatrix}$$
$$= (-1)^{n+i} g(t)\, U\begin{pmatrix} u_0, \ldots, & & \ldots, u_n \\ \bar{t}_0, \ldots, \bar{t}_i, \bar{t}_{i+2}, \ldots, & & \bar{t}_{n+1} \end{pmatrix}$$

The two determinants can be nonnegative iff $g(t) = 0$.

LEMMA 2. *Let $u_0, u_1, \ldots, u_{k+1}$ be continuous functions on the interval I whose interior is $(a, b) \neq I$. If $\{u_i\}_{i=0}^{k}$ is a T-system on I and $\{u_i\}_{i=0}^{k+1}$ is a T-system on (a, b) then it is also a T-system on I.*

Proof. Let $t_0 < t_1 < \cdots < t_{k+1}$ be $k+1$ points of I.

$$U\begin{pmatrix} u_0, u_1, \ldots, u_{k+1} \\ t_0, t_1, \ldots, t_{k+1} \end{pmatrix} \geqslant 0$$

with equality only if $t_0 = a$ or $t_{k+1} = b$, but if the determinant vanishes then by Lemma 1, u_{k+1} is a linear combination of $u_0, u_1, \ldots, u_k$ on some subinterval of I, in contradiction to the assumption on $u_0, u_1, \ldots, u_{k+1}$.

3. TRANSFORMING A T-SYSTEM INTO A CT-SYSTEM

Let M be a real set and $\{u_i\}_{i=0}^{n}$ a T-system on M. Following [2] we define:

DEFINITION 3. A totally ordered set M, is said to have property (D) if (i) for all $x, y \in M$ with $x < y$ there exists an element $z \in M$ with $x < z < y$, and (ii) inf M and sup M do not belong to M.

DEFINITION 4. A totally ordered set M, is said to have property $(\bar{D})$ if (i) it has property (i) of definition 3 and, (ii) M contains at most one of the elements, inf M and sup M.

DEFINITION 5. A k-dimensional space of functions defined on a real set M is called oriented T-space if for every basis $\{f_i\}_{i=1}^k$

$$\operatorname{sgn} U\begin{pmatrix} f_1, f_2, \dots, f_k \\ t_1, t_2, \dots, t_k \end{pmatrix} \text{ is either always } +1 \text{ or always } -1$$

whenever $t_1 < t_2 < \cdots < t_k$ are elements of M. The basis $\{f_i\}_{i=1}^n$ is called oriented T-system.

Let M be a real set having property $(\bar{D})$, we may assume that $b = \sup M \leqslant \infty$, is not in M, if $\{u_i\}_{i=0}^n$ is a T-system on M, we define:

$$w(t) = \operatorname{Max}\{| u_i(t)|; i = 0, 1, \dots, n\}, \qquad t \in M.$$

Clearly, $w(t) > 0$. It can be readily seen that the functions:

$$v_i = \frac{u_i}{w}, \qquad i = 0, 1, \dots, n \tag{4}$$

form a T-system on M and if $\{u_i\}_{i=0}^n$ is oriented, so is $\{v_i\}_{i=0}^n$.

If $\{y_k\}_{k=1}^\infty$ is a sequence in M, converging to b, it contains a subsequence $\{x_k\}_{k=1}^\infty$ such that $\lim_{k\to\infty} v_i(x_k)$ exists for all i. If $a = \inf M \geqslant -\infty$ is in M and if there exists a sequence $\{x_k\}_{k=1}^\infty$ such that $A = (v_0(a), v_1(a), \dots, v_n(a))$ and $(\lim_{k\to\infty} v_0(x_k), \lim_{k\to\infty} v_1(x_k), \dots, \lim_{k\to\infty} v_n(x_k))$ are not proportional, we define:

$$v_i(b) = \lim_{k\to\infty} v_i(x_k), \qquad i = 0, 1, \dots, n, \tag{5}$$

and if $a \notin M$ or, if the limit vectors are always proportional to A we define $v_i(b)$ as in (5), by any sequence $\{x_k\}$ for which the limits exist.

We say that a T-system $\{v_i\}_{i=0}^n$ on M has property (*) in M if $a \notin M$ or $B = (v_0(b), v_1(b), \dots, v_n(b))$ and A are not proportional.

THEOREM 1. *Let $\{u_i\}_{i=0}^n$ be an oriented T-system on a real set M having property $(\bar{D})$ and let $\{v_i\}_{i=0}^n$ be defined on $M \cup \{b\}$ by* (4) *and* (5). *The linear span of $\{u_0, u_1, \dots, u_n\}$ contains an n-dimensional oriented T-space on M iff $\{v_i\}_{i=0}^n$ has property* (*) *in M.*

Proof. (i) *Sufficiency*: The determinants

$$U\begin{pmatrix} v_0, v_1, \dots, v_n \\ t_0, t_1, \dots, t_n \end{pmatrix}$$

are positive whenever $t_0 < t_1 < \cdots < t_n$ are in M. We first show that they are positive with $t_n = b$. By the definition of $v_i(b)$, we have

$$U\begin{pmatrix} v_0, v_1, \dots, v_{n-1}, v_n \\ t_0, t_1, \dots, t_{n-1}, b \end{pmatrix} \geqslant 0. \tag{6}$$

Assume that for some $\bar{t}_0 < \bar{t}_1 < \cdots < \bar{t}_{n-1} < b$ the determinant (6) vanishes. Since $\{v_i\}_{i=0}^n$ is a T-system on M, we have:

$$v_i(b) = \sum_{j=0}^{n-1} a_j v_i(\bar{t}_j) \qquad i = 0, 1, \ldots, n.$$

We call a point $\bar{t}_j$, essential, if $a_j \neq 0$. Since $|v_i(b)| = 1$ for some i, there exists an eseential point $\bar{t}_{j_0}$ and by property (*) we can choose $\bar{t}_{j_0} \neq a$.

Let x, y be two points in M with $\bar{t}_{j_0-1} < x < \bar{t}_{j_0} < y < \bar{t}_{j_0+1}$ (if $j_0 = 0$ set $\bar{t}_{-1} = a$ and if $j_0 = n - 1$ set $t_n = b$).

Span $\{v_0, v_1, \ldots, v_n\}$ contains a function vanishing on $\{\bar{t}_j\}_{j=0}^{n-1}$ and we may assume that this function is v_n.

Since both determinants:

$$U\begin{pmatrix} v_0, \ldots, & & , \ldots, v_n \\ \bar{t}_0, \ldots, & x, \bar{t}_{j_0}, & \ldots, \bar{t}_{n-1} \end{pmatrix}$$

and

$$U\begin{pmatrix} v_0, \ldots, & & , \ldots, v_n \\ \bar{t}_0, \ldots, & \bar{t}_{j_0}, y, & \ldots, \bar{t}_{n-1} \end{pmatrix}$$

are positive:

$$v_n(x) \cdot v_n(y) < 0. \tag{7}$$

Let

$$U = U\begin{pmatrix} v_0, v_1, \ldots, v_{n-1}, v_n \\ \bar{t}_0, \bar{t}_1, \ldots, \bar{t}_{n-1}, b \end{pmatrix}. \tag{8}$$

Define U_x, (U_y) to be the determinant U with x (y) replacing $\bar{t}_{j_0}$.

If $U_x = 0$ we have $v_i(b) = \sum_{j=0}^{n-1} a_j v_i(\bar{t}_j) = \sum_{j \neq j_0} b_j v_i(\bar{t}_j) + b_{j_0} v_i(x)$, which implies

$$U\begin{pmatrix} v_0, \ldots, & & , \ldots, v_n \\ \bar{t}_0, \ldots, & \bar{t}_{j_0-1}, x, \bar{t}_{j_0}, & \ldots, \bar{t}_{n-1} \end{pmatrix} = 0 \tag{9}$$

in contradiction to our assumption on $\{v_i\}_{i=0}^n$. The same argument shows that $U_y \neq 0$. By (7) $U_x \cdot U_y < 0$ while both determinants are positive.

Hence, span$\{v_0, v_1, \ldots, v_n\}$ is an oriented T-space on $M \cup \{b\}$. We now show that it contains an n-dimensional oriented T-subspace (on M).

We may assume (by multiplying the last two rows in (6), if necessary, by -1), that $v_n(b) > 0$. Define $\tilde{v}_i = v_i - (v_i(b)/v_n(b)\, v_n$, $i = 0, 1, \ldots, n-1$, and $\tilde{v}_n = v_n$. Since the determinants (6) are positive, span$\{\tilde{v}_0, \tilde{v}_1, \ldots, \tilde{v}_{n-1}\}$ is an oriented T-space on M and hence, span$\{w\tilde{v}_0, w\tilde{v}_1, \ldots, w\tilde{v}_{n-1}\}$ is an oriented T-subspace of span$\{u_0, u_1, \ldots, u_n\}$ (on M).

(ii) *Necessity*: Assume that A and B are proportional and that $S = \operatorname{span}\{v_0, v_1, \ldots, v_n\}$ contains an n-dimensional oriented T-subspace on M. There exist $n+1$ functions $f_0, f_1, \ldots, f_{n-1}, f_n$ in S such that

$$U\begin{pmatrix} f_0, f_1, \ldots, f_n \\ t_0, t_1, \ldots, t_n \end{pmatrix}$$

and

$$U\begin{pmatrix} f_0, f_1, \ldots, f_{n-1} \\ t_0, t_1, \ldots, t_{n-1} \end{pmatrix}$$

are positive whenever $t_0 < t_1 < \cdots < t_n$ are in M. Since $a \in M$, $f_{i_0}(a) \neq 0$ for some $0 \leqslant i_0 \leqslant n-1$ and since $B \neq 0$ and is proportional to A, $f_{i_0}(b) \neq 0$. Clearly:

$$f_{i_0}(b)\, U\begin{pmatrix} f_0, f_1, \ldots, f_n \\ a, \;\; t_1, \ldots, t_n \end{pmatrix} = (-1)^n f_{i_0}(a)\, U\begin{pmatrix} f_0, \ldots, f_{n-1}, f_n \\ t_1, \ldots, t_n, \;\; b \end{pmatrix} \tag{10}$$

and

$$f_{i_0}(b)\, U\begin{pmatrix} f_0, f_1, \ldots, f_{n-1} \\ a, \;\; t_1, \ldots, t_{n-1} \end{pmatrix} = (-1)^{n-1} f_{i_0}(a)\, U\begin{pmatrix} f_0, \ldots, f_{n-2}, f_{n-1} \\ t_1, \ldots, t_{n-1}, b \end{pmatrix} \tag{11}$$

whenever $a < t_1 < \cdots < t_n < b$ are in M. Since the determinants in the left hand side of (10) and (11) are positive and the determinants on the right are nonnegative, they must be also positive. However, (10) and (11) cannot hold simultaneously.

Remark. It follows that all or none of the limit vectors (5), are proportional to A.

COROLLARY 1. [2] *Let M be a real set having property (D) and R, an $(n+1)$-dimensional oriented T-space on M. Then for $i = 1, 2, \ldots, n$, there exist i-dimensional oriented T-subspaces R_i of R with $R_1 \subset R_2 \subset \cdots \subset R_n \subset R$.*

COROLLARY 2. ([1] *and* [2]). *Let I be an open interval and $u_0, u_1, \ldots, u_n$ be continuous functions on I forming a T-system. Corollary* 1 *holds for* $\operatorname{span}\{u_0, u_1, \ldots, u_n\}$.

THEOREM 2. *Let $u_0, u_1, \ldots, u_n$ be continuous functions on an interval I, forming a CT-system on its interior (a, b). $\{u_i\}_{i=0}^n$ is a CT-system on I iff $u_0 > 0$ on I.*

Proof. The necessity is obvious and by Lemma 2 the condition is also sufficient.

THEOREM 3. *Let M be a real set having the property: if $x, y \in M$ with $x < y$ then there exists an element $z \in M$ with $x < z < y$, and let $\{u_i\}_{i=0}^n$, $n \geqslant 3$, be defined on a set M_1, $M \subset M_1 \subseteq \overline{M}$, continuous on $M_1 \backslash M$ and forming an oriented CT-system on M. It is an oriented CT-system on M_1 iff so is $\{u_0, u_1, u_2\}$.*

Proof. The determinants involved are of order $\geqslant 4$. Suppose that for some k, $3 \leqslant k \leqslant n$:

$$U\begin{pmatrix} u_0,\ldots, & & & ,\ldots, u_k \\ \ldots, x,\ldots, & y,\ldots, & z,\ldots, & t,\ldots \end{pmatrix} = 0$$

with $x < y < z < t$, $x, y, z, t \in \overline{M}$. If $y, z \in M$ then there is some point $s \in M$ between them and if one of them is in $\overline{M} \backslash M$ then there is some point of M between x and t and we can apply the technique of Lemma 2.

EXAMPLE 1. Let $M = (-2, -1) \cup (1, 2)$

$$\begin{aligned} u_0(t) &= 2, & -2 \leqslant t \leqslant -1, \\ &= -t + 3, & 1 \leqslant t \leqslant 2, \\ u_1(t) &= u_0(-t), & t \in \overline{M}. \end{aligned}$$

$\{u_0, u_1\}$ is an oriented CT-system on M but not on $\overline{M}$ although $u_0 > 0$ on $\overline{M}$

EXAMPLE 2. Let $M = (-2, -1) \cup \{0\} \cup (1, 2)$

u_0, u_1, u_2 are defined on $\overline{M}$ by

$$u_0(t) = 1,$$

$$u_1(t) = t,$$

and

$$\begin{aligned} u_2(t) &= t^2, & t \in [-2, -1] \cup [1, 2], \\ &= 1, & t = 0. \end{aligned}$$

$\{u_0, u_1, u_2\}$ is an oriented CT-system on M but not on $\overline{M}$ although $\{u_0, u_1\}$ is.

We conclude by noting that for every n, there exists a T-space on some half-open interval with no T-subspace of codimension 1. For odd dimensional spaces, span$\{1, \cos t, \sin t,\ldots, \cos nt, \sin nt\}$ on $[0, 2\pi)$ provide counter-examples and for even dimension, the span of $f_1(t) = t$ and $f_i(t) = t^{i-2}(t^2 - 1)$ $i = 2,\ldots, n$ on $[-1, 1)$ (see [3]) has no T-subspace of codimension 1, by Theorem 1.

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